A note on "The steadiness of separating meandering currents" by Peter Jan Van Leeuwen and Will P.M. De Ruijter

Doron Nof[1,2], Volodymyr Zharkov[3], Wilton Arruda[1,4], Thierry Pichevin[5], Stephen Van Gorder[1], and Nathan Paldor[6]

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1. Department of Earth, Oceans and Atmospheric Science, Florida State University, Tallahassee, Florida 32306
2. Geophysical Fluid Dynamics Institute, Florida State University, Tallahassee, Florida 32306
3. Center for Oceanic and Atmospheric Prediction Studies, Florida State University, Tallahassee, Florida 32306
4. Instituto de Matemática, Universidade Federal do Rio de Janeiro (UFRJ), Rio de Janeiro, RJ, Brazil
5. Ecoles de Saint-Cyr Coetquidan, Guer, France.
6. Fredy and Nadine Herrmann Institute of Earth Sciences, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904 Israel.

Abstract

Using integration constraints and scale analysis, Van Leeuwen and De Ruijter (2009, VL-DR, hereafter) argue that the downstream flow in the momentum imbalance articles of Nof and Pichevin appearing in the nineties and later on (collectively named NP, by VL-DR) must obey the additional downstream (critical) condition $u^2 = g'h$ (where u is the speed, g' the reduced gravity and h is the thickness). They then further argue that this additional condition provides "a strong limitation on the generality of their results". We argue that, while VL-DR condition of $u^2 = g'h$ is valid for a *purely* geostrophic flow downstream, it is *inapplicable* for the NP solutions because the assumption of a purely geostrophic flow (i.e., $fu = -g'h_y,\ \partial/\partial t = 0,\ and\ v = 0$) was never made at *any* downstream cross-section in NP. Instead, the familiar assumption of a *cross-stream* geostrophic balance in a boundary current, which is slowly varying in the downstream direction, as well as time, has been made (i.e., $fu \approx -g'h_y,\ v \ll u$, small $\partial/\partial t$ but non-zero). Perhaps we originally were not as clear about that as we should have been but this implies that the basic state around which VL-DR expanded their Taylor series does not exist in NP and, consequently, their expansion fails to say anything about NP. In our view, the "strong limitation" that they allude to does not exist.

1. **Introduction**

The easiest way to introduce the NP momentum imbalance idea is via a northward outflow problem (e.g., Pichevin and Nof, 1997, Nof 2005) rather than via a retroflection problem, which is more difficult to understand. (Ironically, this is actually the order that we did the original work but the sometime treacherous road of getting submitted articles to appear reversed the order in which they were published.) In an attempt to make this note at least semi-self-contained, we reproduce below a few figures and equations from the earlier NP articles.

Consider the hypothetical steady northward (inviscid) reduced gravity outflow situation shown in **Fig. 1**. The integrated momentum-flux along the coast (obtained by an integration of the x momentum equation along the contour ABCDA) is,

$$\int_0^L (hu^2 + g'h^2/2 - f\psi)\,dy = 0, \qquad (1)$$

where, u is the speed along the coast, h the thickness, f is the Coriolis parameter, ψ the stream function, $g' = g\Delta\rho/\rho$ (where $\Delta\rho, \rho$ are the density difference /density of the layers and $\Delta\rho/\rho \ll 1$)) and L is the width of the boundary current downstream at CD (i.e., y = L when $h = 0$). Note that symbols and abbreviations are defined in both the text and the Appendix. L, which is a weak function of x, depends on the outflow potential vorticity and is of the order of the Rossby radius,

$$R_D = (2g'Q)^{1/4} / f^{3/4},$$

where Q is the outflow's volume flux (equals $g'H^2/2f$, where H is a depth scale). Assuming steadiness, one dimensionality (i.e., v << u, but non-zero) and geostrophy in the cross-stream direction at CD (but allowing the flow to vary on a much longer downstream length scale, see, e.g., Charney, 1955), and neglecting terms $\sim O(\beta R_D / f_0) \sim (0.01)$, we get,

$$\int_0^L h u^2 dy = 0. \qquad (2)$$

Obviously, (2) cannot be satisfied so there cannot be a steady outflow of the kind pictured in **Fig. 1**. This is what NP termed the "Paradox". Aside from the momentum constraint discussed here, there may also be a, yet unknown, hydraulic constraint involving a condition similar to the familiar criticality of uniform flow alluded to by VL-DR $(u^2 = g'h)$. This would ensure the establishment of stationary waves downstream because their propagation tendency will be arrested by the advection. Whether or not such an additional condition exists is of no consequence to the more general time dependent NP problem discussed below. (Note that the above hypothetically steady problem is defined here as problem # 1, whereas the more general time dependent problem will be defined, in a moment, as problem number 2).

NP resolved the paradox by arguing that a chain of eddies (**Fig. 2**) is formed on the western side of the outflow to compensate for the momentum flux of the jet on the eastern side. This way, the momentum flux of the westward moving eddies balances the momentum expressed by (2) via a nonzero term on the right hand side of (2). Since eddies move westward (due to β) much more slowly than their orbital speed (which is of the same order as the mean flow downstream), they are much larger than R_D (see NP), and so is the x scale of the downstream current.

NP then took the above and applied it to the (southern hemisphere) retroflection case (**Fig. 3**) near a coastline with zero slant $g \to 0$ (so that the coast is zonal). Here, there are *two* currents with their momentum-flux pointing westward (because the momentum is proportional to u^2, not *u*) so the mass flux going into the eddies is larger than in the northward outflow problem and, as a result, the eddies themselves are also larger (see Nof and Pichevin**,** 1996, Pichevin et al. 1999). It is no surprise, therefore, that retroflection eddies are the largest rings in the world ocean (Olson and Evans, 1986; Olson, 1991).

We see that the NP case involves actually two sub problems. The first is a simple northward outflow problem from a point source on a beta plane or its analogous retroflection along a zonal wall. This problem involves a hypothetical eastward coastal current that is initially assumed to be steady and slowly varying in x. This is referred to as problem # 1. We then reject this possibility on the grounds that it does not satisfy the momentum integral, and regard this as a paradox. NP then address the second problem where the steadiness is relaxed to allow for eddies to periodically form and shed on the west side. This is referred to as problem # 2.

VL-DR took a purely geostrophic basic state (i.e., steady, no v), which does not exist in either of these two problems (1 & 2), and say that there is an "additional constraint" to problem # 2. Recall, however, that problem # 2 is an *unsteady* problem so steady considerations do not apply. It will be apparent from Sections 2 and 3 that we agree that there might be some kind of a control (different, however, from the uniform flow condition derived by VL-DR) to problem # 1. By “control” we mean a condition under which the flow (which is varying in x) will support a stationary wave (i.e., the steady advective flow cancels the wave propagation tendency). Since this problem is dismissed as unphysical anyway, we are not sure what is the sense in looking for it. We shall also see that Problem # 2 cannot possibly have such a constraint as it is *unsteady*. That is to say, VL-DR basic state does not exist in NP (in either problem 1 or 2) and, therefore, their expansion and conclusions are irrelevant to both cases.

2) VL-DR argument and its relationship to NP

In their Appendix, immediately below A6, VL-DR correctly argue that, when the flow is zonal, steady and purely geostrophic at any downstream cross-section, then the flow cannot develop meanders nor can it be attached to either meanders or a retroflection upstream or downstream unless $u^2 = g'h$. (Note that VL-DR Taylor series analysis is analogous to that used in Killworth, 1983, and see Pratt and Whitehead 2008 for a discussion of the criticality condition, $u^2 = g'h$). When the flow and/or its boundaries (a front on the southern side in our case) *do vary* with both x and t then neither $v = 0$ nor $\partial/\partial t = 0$ and this criticality condition need *not* be satisfied.

In NP, the downstream flow is merely assumed to be geostrophic across CD (**Figs. 1, 2** and **3**) in agreement with the common assumption made in any slowly varying boundary current (in both x and t). Nowhere has it been assumed, explicitly or implicitly, that, $\partial/\partial t = 0$, $\partial/\partial x \equiv 0$, or $v \equiv 0$ anywhere. It is expected that neither of the three will be identically zero anywhere (except the wall in the outflow problem) because the retroflection eddy is much larger than the downstream current width and its generation period is long $(\beta R_D)^{1}$, but not infinity. Specifically, it is expected that the downstream current will vary on the same larger scale as the eddy as well as on the longer time scale.

Even when the downstream current contains meanders with an amplitude reaching one third of their length, the geostrophic approximation across CD is still valid to the order of ~ $(1/3)^2$ ~ 10% (as our numerical runs confirm, see **Fig. 7a** in Pichevin et al. 1999). Just to be absolutely sure, we also checked whether any of our numerical solutions happen to satisfy the condition $u^2 = g'h$ or something similar to it, which implies *decreasing* velocities toward the front (h $\rightarrow$ 0). We noted that, on the contrary, in all of our dozens of experiments the streamlines appear to be uniformly spaced implying that the velocity *increases* (rather than decreases) toward the front.

3. The mathematical aspect of VL-DR Taylor expansion:

From a mathematical viewpoint, the essence of the argument regarding the variability in x presented in Section 2 is that there exists a small parameter, namely v/u, which we will call ε. (Note that we are speaking here about both problem # 1 and # 2, and that ε here is not the same as that used in NP, $(bR_d^2 / f.)$Equation (2) has to hold only to order e^0, the terms that constitute the "mean state". From this point onward, the problem is clear -- one can expand the terms in the mean state to any of their Taylor series order and they will *not* yield the same effects as those of the order ε terms. Namely, NP are relying on the next order terms (and not on the mean state variables), whereas VL-DR analyze only the mean state variables.

This situation can perhaps be best understood if we examine the example of a long-wave instability (e.g., Killworth et al., 1984, Paldor and Stern 1984): As long as k = 0 (where k is the zonal wave-number that is assumed to be a small parameter), the flow is *stable* and any analysis based on the functions of this state will not produce anything new. Only when one brings in the next order terms (in

k) the instability occurs and some new unstable features of the flow field emerge. Similarly, in the NP case, as long as one remains in an *exact* (imaginable) downstream geostrophic regime (i.e., steady, no v) then the VL-DR arguments hold. However, even the slightest deviation from this state in either x or t will bring in the paradox that NP are alluding to. This easily explains the $u^2 = g'h$ issue, which NP, of course, do not get in their time-dependent simulations, but VL-DR get in their analysis of the mean state.

4) Discussion

In the beginning of their article, VL-DR discuss the mechanism leading to retroflection, noting that NP attribute it to a momentum imbalance of the kind explained above. VL-DR later state: "Although the idea of NP is appealing, it seems to be contradicted by other studies." VL-DR then list Dijkstra and De Ruijter (2001), and Ou and De Ruijter (1983) as evidence for the contradiction of NP with previous articles but never spelled out clearly what the contradiction between these steady problems and NP unsteady solution really is. Ou and De Ruijter (1983) considered different physical systems (i.e., different boundary conditions) than we did so it is not surprising that the results are not the same (see also Pichevin et al. 2009). We have countless counter-examples (to VL-DR statements) where the numerics support our analytics. One example is displayed in **Fig. 8** of Pichevin et al. (1999) and a second is shown in **Fig. 8** of Zharkov et al. (2010). In the second example, the red solid and dashed lines (theory) and red diamonds (numerics) correspond to the NP case ($\gamma = 0$) and those in black color are closely related to VL-DR ($\gamma = 90^{\circ}$).

Finally, we respectfully disagree with the VL-DR statement: "The apparent contradiction between previous work on separating currents, and the more recent work by NP is solved." We think that VL-DR presented interesting arguments but have not resolved any clearly identifiable problem or contradiction. The basic state around which they expanded their Taylor series simply does not exist in NP so their expansion is irrelevant to NP, and there are no additional constraints.

5. Acknowledgements:

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6. Symbols and abbreviations:

u, v - zonal and meridional speeds in the x and y direction

h – upper layer thickness

g'- $g\Delta\rho / \rho$, reduced gravity

f – Coriolis parameter

L(x, t) – downstream current width

Q – outflow or incoming current volume flux

q - outgoing current volume flux

$R_D = (2g'Q)^{1/4} / f^{3/4}$, Rossby radius

ε- small parameter, v/u

β - variation of the Coriolis parameter with latitude

VL-DR—Van Lueewen and De Ruijter (2009)

NP- a series of papers by Nof and Pichevin defined by VL-DR

DDR-- Dijkstra and De Ruijter (2001)

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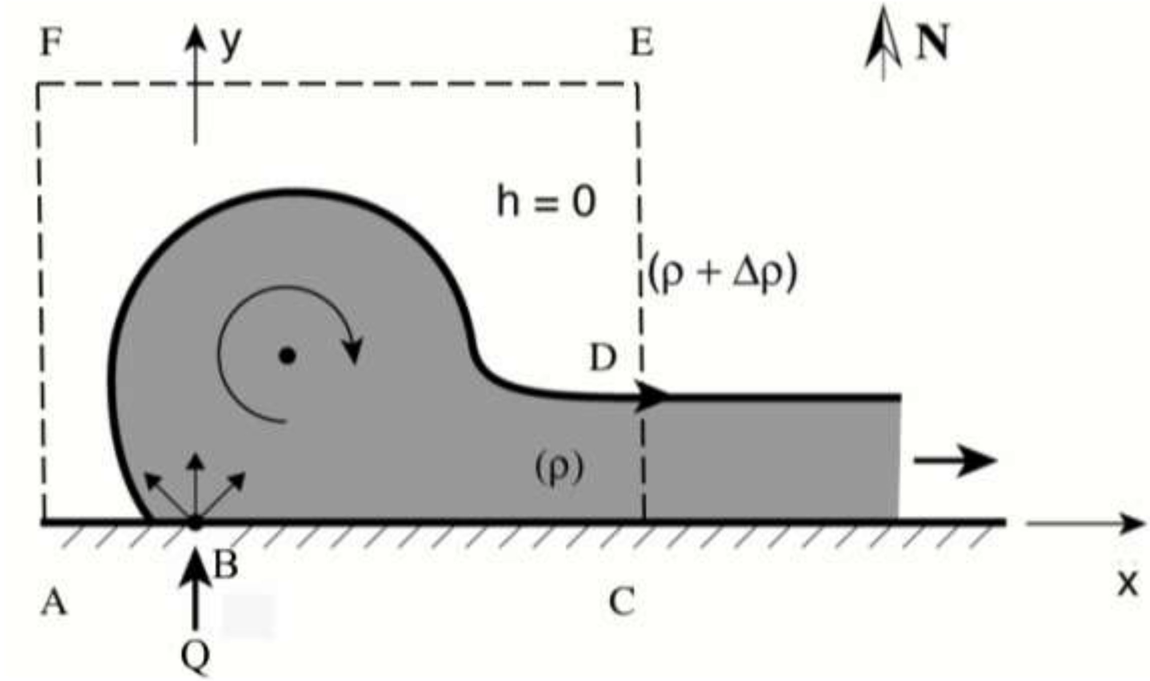

FIG. 1. A schematic diagram of the hypothetical (northern hemisphere) steady configuration shown by NP to be impossible on a β plane (adapted from Nof, 2005). This is because the long-shore momentum flux of the slowly varying downstream boundary current, which is pushing westward, is not balanced. As a result, eddies are periodically shed on the left-hand side (**Fig. 2**).

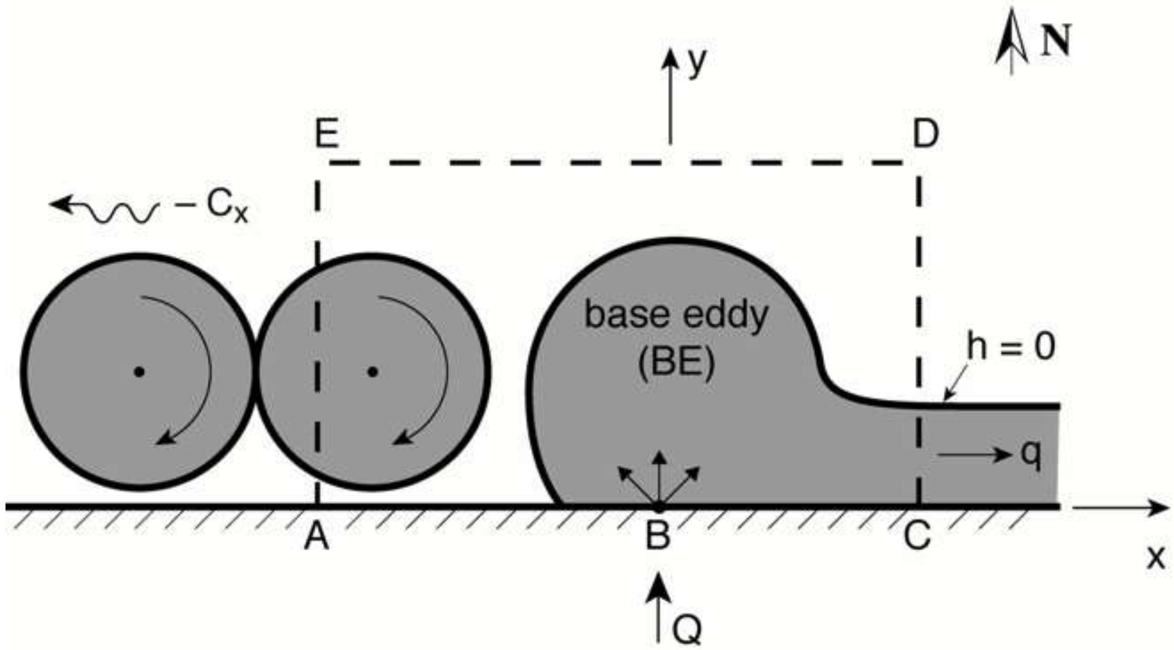

FIG. 2. Schematic diagram showing the PN resolution of the momentum imbalance paradox. “Wiggly” arrow denotes migration. Because of the imbalance shown in **Fig. 1**, anticyclonic eddies are generated on the left-hand side (looking offshore). Through β these eddies are forced to propagate to the left. NP obtained their analytical solution by equating the momentum flux through EA to the momentum flux through CD. The base eddy, which is the eddy in contact with the source, should be distinguished from the already detached eddies downstream.

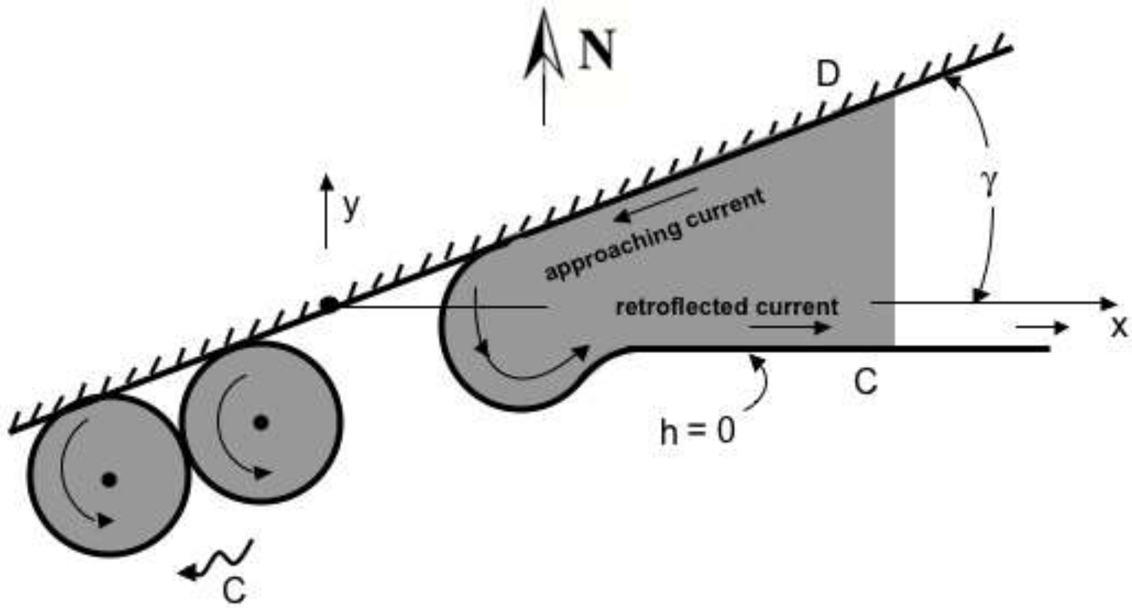

FIG. 3. The Retroflection paradox and its resolution (in the Southern Hemisphere) according to NP (adapted from Pichevin et.al. 1999). To simplify the analysis, NP considered the cases where there is no coastline tilt (i.e., $y \rightarrow 0$). To compensate for the westward momentum flux (or flow force) created by the approaching and retroflected slowly varying boundary currents, westward propagating anticyclonic rings are generated periodically. In this scenario, the eddies exert an eastward momentum flux analogous to the backward push associated with a firing gun. The ‘‘wiggly’’ arrow denotes migration.