

*Geophys. Astrophys. Fluid Dynamics*, Vol. 42, pp. 187-208  
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Printed in Great Britain

# Draining Vortices

DORON NOF

*Department of Oceanography and Geophysical Fluid Dynamics Institute,  
The Florida State University, Tallahassee, Florida 32306-3048, USA*

*(Received 6 August 1987; in final form 21 December 1987)*

A nonlinear barotropic model is used to examine the interaction of isolated vortices (i.e. patches of anomalous vorticity bounded by a free streamline) with vertical walls. The round undisturbed vortices are inviscid and contain cyclonic or anticyclonic circulation; they are embedded in a resting, infinitely broad ocean. The interaction is examined by, conceptually, "cutting" the vortices with a vertical wall at, say,  $t=0$ . The resulting events are studied using a perturbation scheme in  $\varepsilon$ , the nondimensional distance between the wall and the undisturbed edge of the vortices. Analytical solutions for times of  $O(f^{-1})$  (where  $f$  is the Coriolis parameter) are constructed by applying the integrated momentum and continuity equations.

Surprisingly, it is found that, in contrast to classical point vortices which merely translate due to the presence of a wall, regional vortices (i.e. patches of circulation) leak fluid as they interact with the wall. Specifically, when an anticyclone interacts with a wall it leaks fluid from its right side (looking off-shore in the northern hemisphere) whereas a cyclone leaks from its left side. The leaked fluid intrudes along a wall (as a thin jet) by pushing the fluid without vorticity to the side. For times of  $O(f^{-1})$ , the leakage is the only significant feature of the interaction. The width of the jet-like leakage is  $\varepsilon/3$  and the drained mass flux is  $\varepsilon R_0 f r_0^2 H/3$  (where  $H$  is the layer depth,  $r_0$  is the radius of the vortex, and  $R_0$  is a Rossby number).

At large times [ $O(1/\varepsilon f)$ ], the jet-like leakage induces a translation along the wall. It is expected that this translation will be similar to that experienced by classical point vortices (whose orbital speed falls off as  $1/r$ ), but the details are beyond the scope of this study. Ultimately, the leakage ceases and the vortices shrink to a size that barely touches the wall.

KEY WORDS: Vortex dynamics, wall interactions, oceanic eddies.

## 1. INTRODUCTION

When a straight solid wall is suddenly inserted into a barotropic or baroclinic vortex, part of the circulation is blocked due to the impossibility of penetrating the wall. Consequently, the vortex

becomes squashed near the boundary and a complex eddy-wall interaction begins. Such an interaction is an important geophysical problem because ocean eddies frequently encounter continents.

In this paper an attempt is made to examine some of the features that are involved in the process. Specifically, we consider the interaction of a barotropic vortex and a vertical wall on an  $f$ -plane. The barotropic vortex consists of a region with uniform vorticity ( $-2R_0f$ ) corresponding to a cyclonic ( $R_0 > 0$ ) or anticyclonic ( $R_0 < 0$ ) circulation. It is bounded by a vortex sheet beyond which the ocean is stagnant. Namely, the vortex is a *patch* of vorticity; it is very different from the frequently considered point vortex (e.g. see Prandtl and Tietjeus, 1934, pp. 208–213) whose orbital speed falls off as  $1/r$  (where  $r$  is the radius).

The main difference between these two kinds of vortices is that our patch vortex does not extend to infinity as the point vortex does. Instead, it has a *finite* area; this makes it much more applicable to oceanic situations. We shall see that these different features cause the interaction to be fundamentally different in the two cases. In contrast to point vortices which merely translate due to the presence of a solid boundary, isolated eddies leak fluid as they come in contact with a wall. For other studies involving patches of vorticity and "vorticity fronts"\* the reader is referred to Stern and Voropayev (1984), Stern and Pratt (1985), and Nof (1987).

The problem is formulated as follows. Initially, the round barotropic vortex has an orbital speed of  $R_0fr_0$  (where  $R_0$  is a small Rossby number,  $f$  is the Coriolis parameter and a radius  $r_0$ ). Hence, along the bounding streamline (beyond which the ocean is stagnant) the speed is finite ( $R_0fr_0$ ) as is often the case in free streamlines (e.g. see Batchelor, 1968; Garabedian, 1964; Milne-Thompson, 1960). At, say,  $t=0$  a vertical knife-like wall is inserted a distance  $er_0$  from the vortex edge (Figure 1). After an initial period of adjustment [ $O(f^{-1})$ ] the eddy will reach a quasi-steady structure and it is this structure that we shall focus on. We shall see that during this stage the eddy *leaks* fluid along the wall. Anticyclonic eddies leak fluid on the right-hand side (looking off-shore) whereas cyclonic eddies leak on their left side. Application of the Bernoulli integral to the free

\*The term "vorticity front" is used to describe a line along which there is a discontinuous vorticity but a continuous velocity. The edge of our vortices involves a discontinuity in both the velocity and vorticity; hence, it is a vortex sheet.

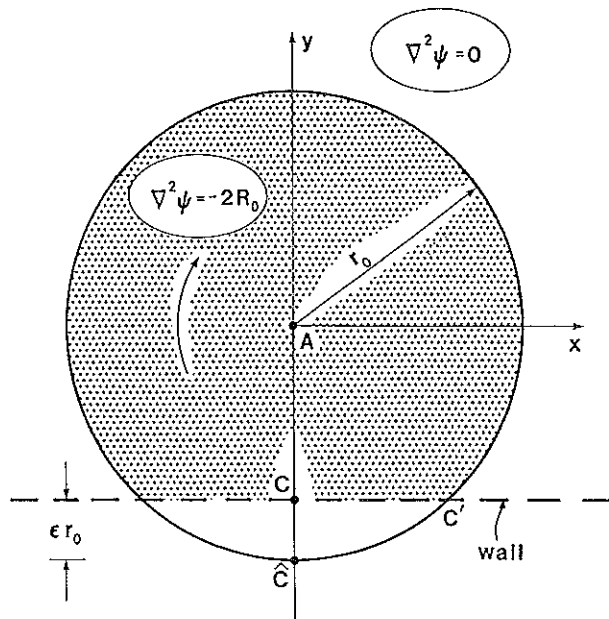


Figure 1 Schematic diagram of the vortex and the wall. Prior to the interaction, the round vortex is stationary and is bounded by a vortex sheet beyond which the fluid is stagnant. At say,  $t=0$ , a straight knife-like wall (dashed line) is inserted a distance  $\epsilon r_0$  from the vortex edge. Note that the initial orbital speed within the eddy is  $-R_0 f r$  so that the relative vorticity is *uniform* and equals  $-2R_0$ . As a result of the wall, a leakage is set up (Figure 2). The streamfunction  $\psi$  is defined by  $\partial\psi/\partial x=v$ ,  $\partial\psi/\partial y=-u$ , where  $u$  and  $v$  are the horizontal velocity components in the  $x$  and  $y$  directions.

bounding streamline indicates that the speed at which the leaked fluid is moving is of the same order as the eddy's orbital speed [ $O(R_0 f r)$  with  $R_0 \ll 1$ ].

Even though the quasi-steady general circulation within the vortex is governed by a *linear* vorticity equation (the so-called Poisson equation which is a second order equation of the elliptic type) the interaction with the wall is *nonlinear*. This results from the fact that the boundary condition is *nonlinear*. Specifically, as we shall see, along the free bounding streamline the square of the velocity must be constant. Furthermore, the position and shape of the free bounding vortex sheet are not known in advance but rather must be found as part of the problem. Because of this difficulty, we shall not even

attempt to find all the details of the interaction. Instead, we shall use the integrated equations of motion which permit the computation of the leakage without knowing all the details of the flow.

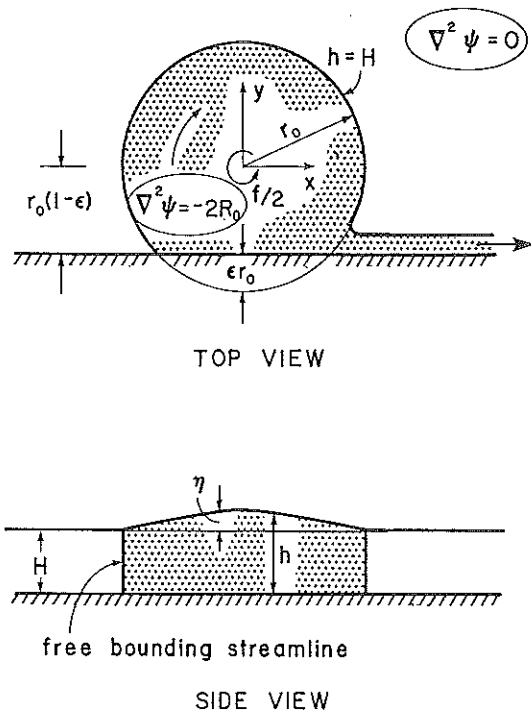
We shall see that such an approach does not neglect nonlinearity and yet enables one to solve for the most important aspects of the problem. The approach relies heavily on the fact that, in the immediate vicinity of the wall, the flow must be one dimensional so that it is also geostrophic. As mentioned, it is taken into account that for times of  $O(f^{-1})$  the problem can be considered to be steady due to the relatively small amount of mass that is leaked.

The paper is organized as follows. The details of the formulation are presented in Section 2 and the general solution in Section 3. Section 4 contains the equations which connect the various regions, and Sections 5 and 6 include the solution and its analysis. Section 7 summarizes this work.

## 2. FORMULATION

*a) General* As briefly mentioned, the eddy-wall interaction is formulated in terms of a knife-like wall that is inserted into the vortex at  $t=0$ . That is, as shown in Figures 1 and 2, we conceptually view the interaction as being the result of an infinitely long wall that cuts the vortex into two sections. The dynamics of the small section (unshaded area in Figure 1) are of no interest to us and we shall focus on the behavior of the larger section (shaded area in Figure 1). Due to the insertion of the wall, part of the circulation within the vortex is blocked (Figure 2). It is expected that this blocked flow (i.e. the original flow corresponding to section  $C\hat{C}$  in Figure 1) will be compensated for by an increase in the speed in section  $AC$  (Figure 1) (i.e. section 2 in Figure 3) and a leakage along the wall (Figure 2). As we shall see, the system goes through an adjustment process which lasts a time of  $O(f^{-1})$ ; it then settles into a quasi-steady balance.

The origin of our coordinate system is located at the center of the eddy; the  $x$  and  $y$  axes are oriented along and across the wall and the system rotates uniformly at  $f/2$  about the  $z$  axis. The undisturbed orbital speed is  $-R_0 f r$ , where  $R_0$  is much smaller than unity and can be either positive (anticyclonic) or negative (cyclonic).



**Figure 2** A sketch of the eddy-wall interaction. Due to the presence of the wall, the anticyclonic vortex leaks fluid on the right-hand side (looking off-shore). As shown in the text, for times of  $O(f^{-1})$ , the problem can be considered steady so that the fluid with anomalous vorticity (shaded) is governed by the so-called Poisson equation ( $\nabla^2\psi = -2R_0$ ). It is bounded by a vortex sheet beyond which the ocean is stagnant ( $\nabla^2\psi = 0$ ). Here,  $h$  is the total depth (i.e.  $h = H + \eta$ , where  $H$  is the undisturbed depth, and  $\eta$  is the free surface vertical displacement which is measured upward from the undisturbed level).

Specifically,  $R_0$  is of the same order as  $\epsilon$ , the nondimensional distance between the wall and the eddy's edge (Figures 1 and 2). The model is hydrostatic and inviscid but the interaction area is not constrained to be quasi-geostrophic in the sense that the *local* Rossby number is not necessarily small.

We shall invoke the rigid lid approximation [i.e. the free surface vertical displacement ( $\eta$ ) obeys  $\eta \ll H$ , where  $H$  is the undisturbed



nose. It results from the fact that in a steady inviscid model there is no mechanism by which momentum can be transferred from the vortex to the outer region.

*b) Scaling* Because of the time dependent aspects of the problem, it is necessary to introduce the scaling at this stage. With the aid of this scaling we can demonstrate that the problem is quasi-steady and, hence, save ourselves unnecessary developments. Contrary to intuition, the scaling is not at all trivial and, for this reason, detailed explanations are provided below. The reader who still finds some of these details to be too complex is advised to continue reading and return to these points later.

As mentioned, we shall now focus our attention on the time scale and show that for  $t \sim O(f^{-1})$  the problem can be taken to be steady. To show this, we note a few aspects of the flow. First, we note that the mass flux blocked by the insertion of the wall is of  $O(\varepsilon^2 f r_0^2 H)$ . Hence, the leaked mass flux is also  $O(\varepsilon^2 f r_0^2 H)$ . This is consistent with a leaked flow width of  $(\varepsilon r_0)$  and a speed of  $O(\varepsilon f r_0)$  because application of the Bernoulli integral along the free bounding streamline speed shows that the leakage speed is of the same order as the eddy orbital speed along the edge. Secondly, we note that the time required for the depletion of the eddy's rim [i.e. the outer ring defined by  $r_0 \leq r \leq (1-\varepsilon)r_0$  which corresponds to the fluid that is in direct contact with the wall] is  $O(1/\varepsilon f)$ . This is the same time scale as that of the circulation within the vortex.

Thirdly, we note that in section 2 (see Figure 3) the effect of the wall penetrates a distance  $\delta \sim O(\varepsilon^{1/2} r_0)$  away from the wall and that the perturbed velocities near the wall ( $u'_2$ ) are  $O(\varepsilon^{3/2} f r_0)$ . Although it is not *a priori* obvious, these scales result from the fact that the integrated mass and momentum must balance so that,

$$\int_0^\delta u'_2 d\xi \sim O(\varepsilon^2 f r_0) \quad \text{and} \quad \int_0^\delta \bar{u}_2 u'_2 d\xi \sim O(\varepsilon^3 f^2 r_0^3),$$

where  $\xi$  measures the distance from the wall and  $\bar{u}_2$  is the mean undisturbed flow in section 2 [ $\sim O(\varepsilon f r_0)$ ]. It will become clear later that the above integrated balances are a direct result of the integrated mass and momentum equations. The right-hand side of the first relationship represents the mass flux of the leakage, whereas the

right-hand side of the second represents the momentum flux of the leakage. Since  $\bar{u}_2 \sim O(\epsilon f r_0)$ , these relationships show that  $u'_2$  can only be of  $O(\epsilon^{3/2} f r_0)$  and  $\delta$  can only be of  $O(\epsilon^{1/2} r_0)$ . It should also be pointed out that the Laplace equation implies that the  $x$  and  $y$  perturbations are of the same order. This suggests that  $CC'$  (Figure 3) is of the same order as  $BC$  indicating that  $\delta \sim O(\epsilon^{1/2} r_0)$ , as stated above.

In the view of the above scales, we may expand all the variables in the form,

$$u^* = \epsilon u^{(0)}(x^*, y^*) + \epsilon^{3/2} u^{(1)}(x^*, y^*, \epsilon t^*) + \epsilon^2 u^{(2)}(x^*, y^*, \epsilon t^*) + \dots$$

$$v^* = \epsilon v^{(0)}(x^*, y^*) + \epsilon^{3/2} v^{(1)}(x^*, y^*, \epsilon t^*) + \epsilon^2 v^{(2)}(x^*, y^*, \epsilon t^*) + \dots$$

$$\psi^* = \epsilon \psi^{(0)}(x^*, y^*) + \epsilon^{3/2} \psi^{(1)}(x^*, y^*, \epsilon t^*) + \epsilon^2 \psi^{(2)}(x^*, y^*, \epsilon t^*) + \dots,$$

where  $u$  and  $v$  have been nondimensionalized by  $\alpha f r_0$  [where  $\alpha = R_0/\epsilon$  and  $R_0 \sim O(\epsilon)$ ],  $\psi$  by  $\alpha f r_0^2$ ,  $x$  and  $y$  by  $r_0$ , and  $t$  by  $f^{-1}$ . As usual, the asterisks (\*) indicate that the variables are nondimensional.

For  $t^* \sim O(1)$ , the expansions can be further simplified (by using a Taylor series) to,

$$u^* = \epsilon u^{(0)}(x^*, y^*) + \epsilon^{3/2} u^{(1)}(x^*, y^*, 0) + \epsilon^2 u^{(2)}(x^*, y^*, 0) \\ + \epsilon^{5/2} t^* \partial u^{(1)}(x^*, y^*, 0) / \partial t^* + \dots$$

This expansion shows that for  $t^* \sim O(1)$ , the largest time dependent terms are of  $O(\epsilon^{5/2})$  illustrating that up to  $O(\epsilon^2)$  the problem can be taken to be *steady*, as stated earlier.

Note that the time dependent motions correspond to a jet-like acceleration along the wall. Clearly, as can be seen from Figure 3, the momentum of the leaked flow must cause an acceleration and movement along the wall. For an anticyclone, this motion is oriented along the negative  $x$ -direction; in a way it is similar to the so-called "image" translation associated with a point vortex situated near a wall. Since at  $t \sim O(f^{-1})$  it is very small [i.e. it is at the most of  $O(\epsilon^{5/2})$ ], its detailed solution is beyond the scope of this study. Indeed, this neglected time dependent motion becomes important only after a very long time [ $O(1/\epsilon f)$ ].

c) *The flow in section 1* In analogy with the classical assumption of "no upstream influence", it is assumed here that section 1 (see Figure 3) remains unaltered. Since the circulation within the vortex is  $O(\varepsilon f r_0)$ , it would take a time of  $O(1/\varepsilon f)$  (after the insertion of the wall) for the information to reach it via advection. Hence, it is appropriate to assume that at times of  $O(f^{-1})$  the flow in section 1 is not influenced by the presence of the wall.

### 3. GENERAL SOLUTION FOR SECTIONS 2 AND 3 (FIGURE 3)

In this section we shall discuss the general solution for the different areas, keeping in mind that our problem is steady because we are not interested in high order approximations [i.e.  $O(\varepsilon^{5/2})$ ].

a) *Section 2* The flow in this section (i.e. section BC in Figure 3) is one dimensional due to the wall [ $v=0$  along  $y=-r_0(1-\varepsilon)$ ]; the section off the wall (i.e. section AB in Figure 3) is undisturbed and remains two dimensional (i.e.  $u^* = \varepsilon y^*$ ;  $v^* = -\varepsilon x^*$ ). The pressure and velocity of the two sections are matched at B (see Figure 4).

Since in the immediate vicinity of the wall  $v \rightarrow 0$ , the governing vorticity equation is

$$\partial u_2^* / \partial y^* = 2\varepsilon, \quad (3.1)$$

giving

$$u^* = 2\varepsilon y^* + A, \quad (3.2)$$

where  $A$  is an integration constant to be determined. The matching of velocity at  $B$  implies that

$$\begin{aligned} \text{(near wall)} \quad u_2^* &= \varepsilon y^* + \varepsilon \underbrace{[y^* + (1-\varepsilon) - \delta^*]}_{O(\varepsilon^{1/2})}; \\ &-(1-\varepsilon-\delta) \leq y^* \leq -(1-\varepsilon); x^* = 0, \end{aligned} \quad (3.3a)$$

$$\text{(off the wall)} \quad u^* = \varepsilon y^*; \quad -(1-\varepsilon) \leq y^* \leq 0; x^* = 0, \quad (3.3b)$$

where the, yet unknown, nondimensional thickness of the boundary

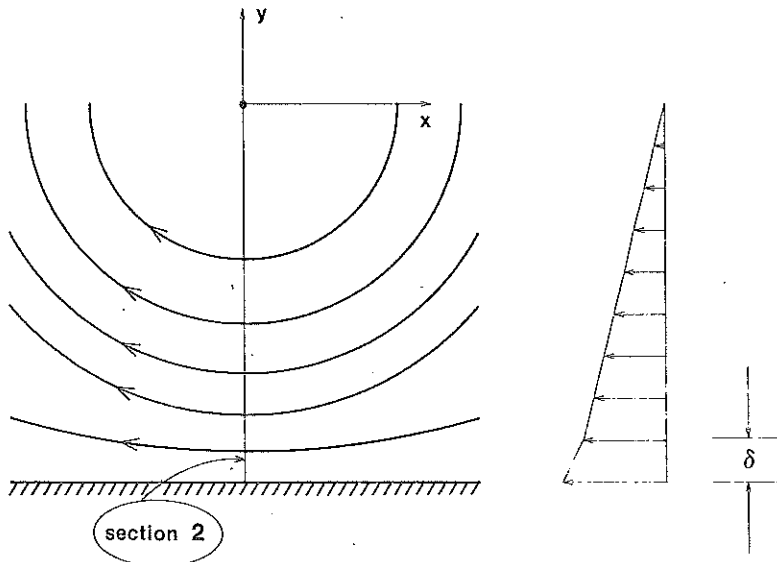


Figure 4 A sketch of the flow in the neighborhood of section 2. In the immediate vicinity of the wall there is a boundary layer and the flow is one dimensional; away from the wall the flow is circular and two dimensional. For  $x=0$ , the velocity and pressure of the two different sections are matched at  $y = -r_0(1-\varepsilon) + \delta$ . The streamlines are shown on the left and the corresponding velocity profile on the right.

layer ( $\delta^*$ ) is defined by  $\delta^* = \delta/r_0$ . Note that the first term in (3.3a) corresponds to the undisturbed state whereas the second is the perturbation due to the wall. As mentioned,  $\delta^* \sim y^* \sim O(\varepsilon^{1/2})$  so that the second term is small as required. Note that, so far, *no terms* have been neglected due to the smallness of  $\varepsilon$  and  $R_0$ .

With the aid of (3.3), we shall now discuss the relationship between the streamfunction ( $\psi_2$ ) and the pressure ( $\eta_2$ ). Since the flow in section 2 is geostrophic, it follows immediately from the  $y$  momentum equation that,

$$\psi_2^* = \eta_2^* + A'. \tag{3.3c}$$

Here,  $A'$  is an integration constant to be determined from the matching condition (pressure, velocity and, of course, streamfunction) at B. Since the flow at any  $r \leq r_0(1-\varepsilon-\delta^*)$  is identical to that

of the undisturbed vortex which obeys the nonlinear momentum equation

$$\alpha u^* \partial v^* / \partial x^* + \alpha v^* \partial v^* / \partial y^* - u^* + \partial \eta^* / \partial y^* = 0$$

and

$$\psi_u^* = \eta_u^* + \frac{1}{2} \alpha [1 - (y^*)^2], \quad (3.3d)$$

it follows that

$$A' = \alpha [\delta^* (1 - \frac{1}{2} \delta^*) + \varepsilon (1 - \frac{1}{2} \varepsilon)].$$

Here, the subscript "u" denotes association with the undisturbed (no wall) vortex. To avoid any possible confusion regarding the definition of the basic state, it should perhaps be stressed again that the basic state corresponds to a situation where there is no eddy-wall interaction, i.e. the eddy is merely "kissing" the wall. The basic state does not correspond to a situation where there is no flow within the vortex.

We finally have

$$\psi_2^* = \eta_2^* + \alpha [\delta^* (1 - \frac{1}{2} \delta^*) + \varepsilon (1 - \frac{1}{2} \varepsilon)]. \quad (3.3e)$$

b) *Section 3* Here, the flow is again one dimensional and geostrophic due to the wall. Hence,

$$u_3^* = 2\varepsilon y^* + B, \quad (3.4)$$

where  $B$  is an integration constant to be determined from the boundary conditions. Application of the Bernoulli integral  $[\frac{1}{2}(u^2 + v^2) + g\eta = F(\psi)]$  along section FE of the free bounding streamline (Figure 3) shows that

$$u_3^* = \varepsilon; \quad y^* = -(1 - \varepsilon) + \gamma^*, \quad (3.5)$$

where  $\gamma^*$  is the, yet unknown, nondimensional width of the leaked flow. It is important to realize that (3.5) is valid because application of the Bernoulli integral along the free bounding streamline shows that if  $\frac{1}{2}(u^2 + v^2)$  is constant along the streamline then  $\eta$  must also be

constant; this satisfies the requirements for the continuity of pressure across the free streamline (e.g. see Batchelor, 1968; Garabedian, 1964; Milne-Thompson, 1960). A combination of (3.5) and (3.4) gives

$$u_3^* = 2\varepsilon[(y^* + \frac{3}{2}) - \varepsilon - \gamma^*]. \quad (3.6)$$

With the aid of (3.6) we can now derive the relationship between  $\psi_3^*$  and  $\eta_3^*$ . Because the flow is geostrophic, integration of the  $y$  momentum equation gives

$$\psi_3^* = \eta_3^* + \text{constant},$$

where, as before, the constant is to be determined from the boundary conditions. Recall that  $\psi^*$  was defined to be zero along the bounding streamline where  $\eta_3^* = 0$ . Hence, the constant must vanish and we have

$$\psi_3^* = \eta_3^*. \quad (3.7)$$

*c) General comments* Our solution for sections 2 [relation (3.3a)] and 3 [relation (3.6)] involves two unknowns, the thickness of the boundary layer  $\delta^*$  and the width of the leaked flow  $\gamma^*$ . Two additional equations are, therefore, needed to complete the solution. We shall see in the next section that these additional relations are provided by an integration of the momentum and continuity equation over the shaded area shown in Figure 3.

#### 4. THE CONNECTION BETWEEN THE FLOWS IN THE VARIOUS REGIONS

*a) Continuity* Integration of the continuity equation over the area(s) shown in Figure 3 gives,

$$\iint_S (\partial u^*/\partial x^* + \partial v^*/\partial y^*) dx^* dy^* = \oint u^* dy^* - \oint v^* dx^* = 0, \quad (4.1)$$

where we have used Stokes' theorem and the arrowed circles indicate that the integration is done in a counterclockwise manner. Since

along any streamline  $u dy = v dx$ , (4.1) gives

$$\int_F^G (u^* dy^* - v^* dx^*) + \int_B^C u^* dy^* + \int_D^E u^* dy^* = 0. \quad (4.2)$$

b) *Momentum* In a similar fashion to the continuity equation, integration of the (nondimensional) x momentum equation gives

$$\iint_s [\alpha u^* (\partial u^* / \partial x^*) + \alpha v^* (\partial u^* / \partial y^*) - v^* + (\partial \eta / \partial x^*)] dx^* dy^* = 0, \quad (4.3)$$

where  $\eta$  has been scaled by  $\alpha f^2 r_0^2 / g$ . By using the continuity equation ( $\partial u^* / \partial x^* + \partial v^* / \partial y^* = 0$ ), (4.3) can also be written in the form

$$\iint_s \left\{ \frac{\partial}{\partial x^*} [\alpha (u^*)^2] + \frac{\partial}{\partial y^*} (\alpha u^* v^*) - \frac{\partial \psi^*}{\partial x^*} + \frac{\partial \eta^*}{\partial x^*} \right\} dx^* dy^* = 0. \quad (4.4)$$

With the aid of Stokes' theorem, (4.4) can be expressed as

$$\oint \alpha (u^*)^2 dy^* - \oint \alpha u^* v^* dx^* - \oint \psi^* dy^* + \oint \eta^* dy^* = 0,$$

which (by noting that  $\eta^* = \psi^* = 0$  along the edge and that  $v^* = 0$  at the wall) can be easily simplified to

$$\begin{aligned} & \int_B^C [\alpha (u_2^*)^2 - \psi_2^* + \eta_2^*] dy^* + \int_D^E [\alpha (u_3^*)^2 - \psi_3^* + \eta_3^*] dy^* \\ & + \int_F^G [\alpha (u_1^*)^2 - \psi_1^* + \eta_1^*] dy^* - \int_F^G \alpha u_1^* v_1^* dx^* \\ & + \int_G^B [\alpha (u^*)^2 - \psi^* + \eta^*] dy^* - \int_G^B \alpha u^* v^* dx^* = 0. \end{aligned} \quad (4.5)$$

Equation (4.5) can be further simplified by considering (3.3d), (3.3e) and (3.7) which give

$$\int_B^C [(u_2^*)^2 + \delta^* (\frac{1}{2} \delta^* - 1) + \epsilon (\frac{1}{2} \epsilon - 1)] dy^* + \int_D^E (u_3^*)^2 dy^*$$

$$\begin{aligned}
& + \int_F^G [(u_u^*)^2 + \delta^*(\frac{1}{2}\delta^* - 1) + \varepsilon(\frac{1}{2}\varepsilon - 1)] dy^* - \int_F^G u_u^* v_u^* dx^* \\
& + \int_G^B [(u_u^*)^2 + \delta^*(\frac{1}{2}\delta^* - 1) + \varepsilon(\frac{1}{2}\varepsilon - 1)] dy^* - \int_G^B u_u^* v_u^* dx^* = 0. \quad (4.6)
\end{aligned}$$

Since the flow in section 1 and section GB remains unaltered, and in the basic undisturbed state (no wall)  $\psi_u^* = \eta_u^*$  along  $\hat{C}C'F$ , the integrals over FGB can be replaced by

$$\alpha^{-1} \int_C^B [\alpha(u_u^*)^2 - \psi_u^* + \eta_u^*] dy^*$$

and this ultimately gives

$$\int_B^C (u_2^*)^2 dy^* + \int_D^E (u_3^*)^2 dy^* + \int_C^B (u_u^*)^2 dy^* = 0. \quad (4.7)$$

A few comments should be made regarding (4.7). First, note that this equation states that the *flow-forces* associated with sections 1, 2 and 3 are balanced. Secondly, note that this balance is similar to that of a jet impinging on a wall. The aspects of such nonrotating interactions have been known for many years (e.g. see Milne-Thomson, 1960) and a rotating jet interaction has been recently studied by Whitehead (1985). Thirdly, recall that, so far, no approximations have been made save the neglect of time dependent motions. As we shall shortly see, Eqs. (4.2) and (4.7) provide the necessary relationship for the computation of our two unknowns,  $\delta^*$  and  $\gamma^*$ .

## 5. SOLUTION

To obtain the detailed solution, all the dependent variables are expanded in power series,

$$u_2^* = \varepsilon u_2^{(0)}(x^*, y^*) + \varepsilon^{3/2} u_2^{(1)}(x^*, y^*) + \dots,$$

$$v_2^* = \varepsilon v_2^{(0)}(x^*, y^*) + \varepsilon^{3/2} v_2^{(1)}(x^*, y^*) + \dots,$$

$$\begin{aligned}
 u_3^* &= \varepsilon u_3^{(1)}(x^*, y^*) + \varepsilon^{3/2} u_3^{(2)}(x^*, y^*) + \varepsilon^2 u_3^{(3)}(x^*, y^*) + \dots, \\
 \gamma^* &= \varepsilon \gamma^{(1)} + \varepsilon^{3/2} \gamma^{(2)} + \dots, \\
 \delta^* &= \varepsilon^{1/2} \delta^{(1)} + \varepsilon \delta^{(2)} + \varepsilon^{3/2} \delta^{(3)} + \dots,
 \end{aligned}
 \tag{5.1}$$

where the terms  $u^{(0)}$  and  $v^{(0)}$  correspond to the basic undisturbed eddy, i.e. an eddy that does not interact with a wall. Note that the terms involving  $u^{(0)}$  and  $v^{(0)}$  contain  $\varepsilon$  because even the motions in our basic state are weak [i.e.  $R_0 \sim O(\varepsilon)$ ]. In terms of our general solutions for sections 2 and 3, the expansions are

$$u_2^* = \varepsilon y^* + \varepsilon \underbrace{[y^* + 1 - \varepsilon^{1/2} \delta^{(1)}]}_{O(\varepsilon^{1/2})} + \dots,
 \tag{5.2a}$$

$$u_3^* = 2\varepsilon(y^* + \frac{3}{2}) - 2\varepsilon^2(1 + \gamma^{(1)}) + \dots.
 \tag{5.2b}$$

Substituting (5.2) into the integrated continuity Eq. (4.2) and keeping only the highest order terms [ $O(\varepsilon^2)$ ] gives

$$\gamma^{(1)} - 1 + \frac{1}{2}(\delta^{(1)})^2 = 0.
 \tag{5.3}$$

Similarly, substituting (5.2) into the integrated momentum Eq. (4.7) and keeping the highest order terms [ $O(\varepsilon^3)$ ] gives

$$\gamma^{(1)} + 1 = (\delta^{(1)})^2.
 \tag{5.4}$$

The solution of (5.3) and (5.4) is

$$\gamma^{(1)} = \frac{1}{3}, \quad \delta^{(1)} = 2(3)^{-1/2}.
 \tag{5.5}$$

It is important to note that the continuity Eq. (4.2) and the momentum Eq. (4.7) could not have been simultaneously balanced with any scaling (for  $\delta$ ) that is different from  $\varepsilon^{1/2} r_0$ , that is, the thickness of the boundary current in section 2 can *only* be of  $O(\varepsilon^{1/2} r_0)$ , as stated earlier.

In dimensional form, our complete solution is

$$\gamma = \frac{1}{3}\varepsilon r_0 + O(\varepsilon^{3/2}r_0) + \dots, \quad (5.6a)$$

$$\delta = 2r_0(3)^{-1/2}\varepsilon^{1/2} + O(\varepsilon r_0) + \dots, \quad (5.6b)$$

$$u_2 = R_0 f y + R_0 f [y + r_0 - 2r_0(3)^{-1/2}\varepsilon^{1/2}] + O(\varepsilon^2 f r_0) + \dots, \quad (5.6c)$$

$$u_3 = 2R_0 f (y + \frac{2}{3}r_0) - \frac{8}{3}R_0 \varepsilon f r_0 + O(\varepsilon^{5/2} f r_0) + \dots, \quad (5.6d)$$

where, as stated before,  $R_0 \sim O(\varepsilon)$ , and the eddy's undisturbed orbital speed ( $v_\theta$ ) is  $v_\theta = -R_0 f r$ .

For an anticyclonic vortex ( $R_0 > 0$ ) the leakage is on the right-hand side (looking off-shore), as shown in the top panel of Figure 5. For a cyclonic vortex ( $R_0 < 0$ ), on the other hand, the leakage is on the left-hand side as shown in the lower panel of Figure 5.

## 6. DISCUSSION

*a) General remarks* The solution derived in Section 5 is valid only for  $t \sim O(f^{-1})$ . At larger times [e.g.  $t \sim O(1/\varepsilon f)$ ], the translations along the wall (in the negative  $x$ -direction for an anticyclone and in the positive for a cyclone) become important and cannot be neglected. It is expected that ultimately, as  $t \rightarrow \infty$ , the leakages will be terminated so that the final state will include a smaller vortex that is merely "kissing" the wall and an infinitely long strip containing eddy water (Figure 6). Note that, by analogy to a rocket, the eddy is expected to translate along the wall even though the jet is completely expelled. Using the integrated momentum equation, it is possible to show that there is no steady solution for this state; the long narrow strip must always have a propagating front.

The reader may wonder whether the region on the left-hand side of the interaction area (shown in Figure 3) involves an entrainment of irrotational fluid analogous to the detrainment on the right-hand side. Dynamically, there is really no reason to expect such an intrusion because the downstream of the separation point (which is not necessarily radially symmetric) is free to adjust to its associated upstream structure (i.e. section 2). In any event, our inviscid model

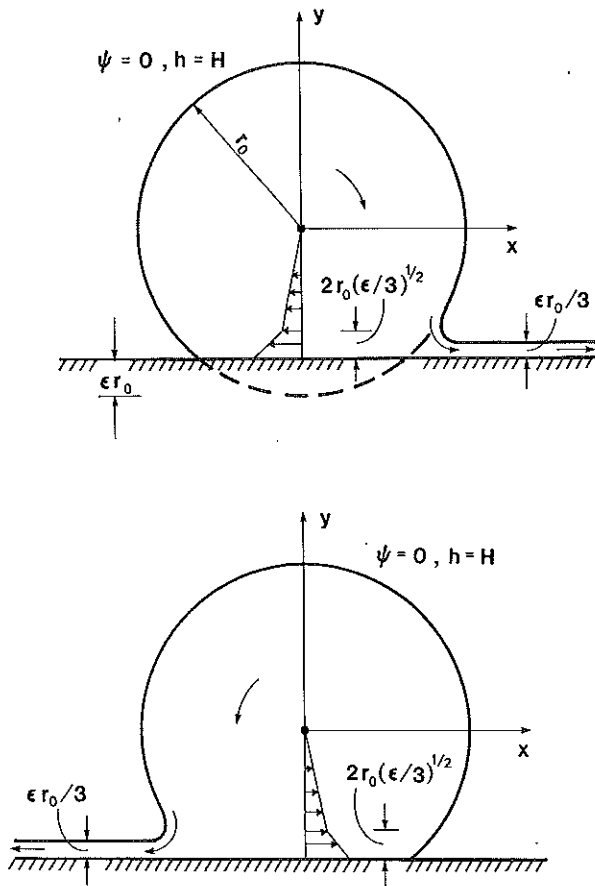


Figure 5 The predicted leakage for an anticyclonic eddy (upper panel) and a cyclonic eddy (lower panel) that are interacting with a wall. The width of the leakage is  $\epsilon r_0/3$  and the thickness of the boundary layer near the wall is  $2r_0(\epsilon/3)$ .

cannot handle such an entrainment because it has no mechanism for altering the vorticity of the surrounding fluid.

It should be pointed out that the area of the integration section shown in Figure 3 is of  $O(\epsilon r_0^2)$ . Hence, the contribution of the neglected time dependent terms, originating from  $\partial u^*/\partial t^*$ , to the integrated momentum equation is of  $O(\epsilon^{7/2})$  which is a higher power than the largest order terms  $O(\epsilon^3)$ , as should be the case. However,

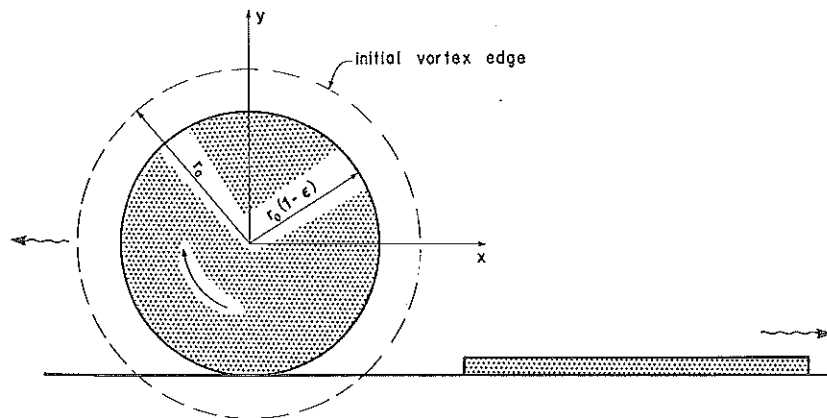


Figure 6 The expected ultimate stage of the eddy-wall interaction. The outer rim of the (anticyclonic) eddy has leaked out and has formed a long narrow strip on the right-hand side. As a result, the vortex has shrunk to a radius of  $r_0(1-\epsilon)$  and is merely touching the wall. It is expected that such a situation will be established after a (relatively) long time [ $t \sim O(\epsilon f)^{-1}$ ].

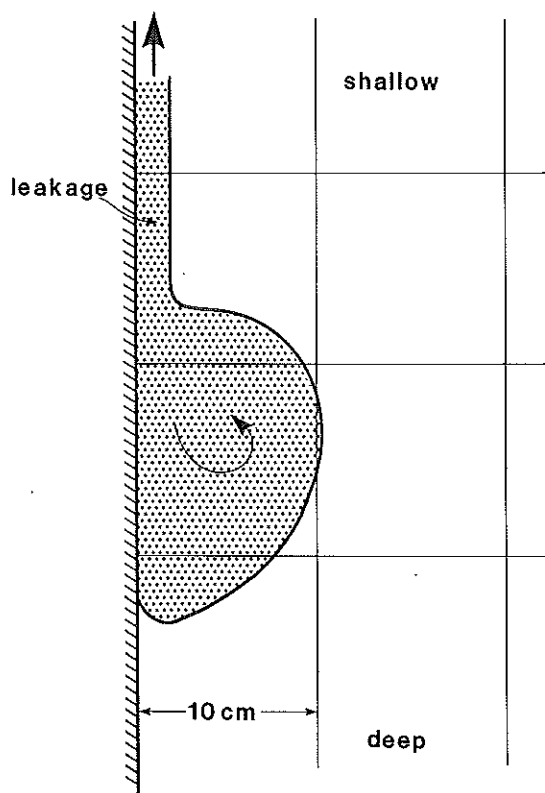
an integration over the whole vortex would not be appropriate because the term in question would be of  $O(\epsilon^{5/2})$  which is larger than the terms that are kept  $O(\epsilon^3)$ .

In addition, it is worth mentioning that the outer streamline connecting the eddy to the leakage turns a corner whose length scale is of  $O(\epsilon r_0)$  (Figure 3). Since the speed is slow,  $O(\epsilon f r_0)$ , the centrifugal acceleration ( $v^2/r$ ) is of  $O(\epsilon f^2 r_0)$  and any unusual behavior is not expected.

Finally, the fact that the cyclonic intrusion propagates along the left wall may appear, at first, to be counterintuitive because the intrusion seems to be shallower than the outer fluid. As shown in Nof (1987), a close examination of the area in the vicinity of the intrusion's nose illustrates that the head of the intrusion is actually higher than the fluid ahead of it and that, consequently, there is no difficulty with a cyclonic intrusion propagating along a wall [see Figure 6 in Nof (1987)].

*b) Comparison to other studies* There are two independent studies that observed phenomena similar to that predicted by our analytical solution. First, the laboratory experiments of Agnon (1986) indicate

that when cyclonic rings interact with a meridional wall they leak fluid on their left side, as predicted by our solution. The leakage is taking place in the form of a thin one dimensional jet, as suggested by the analysis presented earlier (Figure 7). Since Agnon's experiments were conducted on a sloping bottom, the eddies were always forced against the wall (by the slope-induced westward drift) and, hence, our ultimate state of an eddy that is merely kissing the wall



**Figure 7** A diagram of a typical interaction between a barotropic cyclone and a vertical wall. The diagram was adapted (by eye) from a photograph of a laboratory experiment conducted by Agnon (1986); the eddy's boundary was identified by tracing the maximum gradient of the dye in (his) Figure IIb. Note that other experiments conducted by Agnon (1986) displayed a very similar structure, i.e. there is leakage via a thin jet on the left-hand side (looking off-shore). The observed leakage should be compared to the predicted draining shown in the lower panel of Figure 5.

could not have been reached. It should be pointed out that Agnon observed weak leakages even away from any boundaries. These leakages were, however, very small compared to the wall-induced jets. So much so that, for intense eddies, the leakage became visible only when the eddy interacted with the wall.

The other study which observed leakage due to interaction with boundaries is that of Chapman and Brink (1987) who examined the interaction with a slope. Their linear numerical model shows that anticyclonic eddies leak fluid on their right side when interacting with a slope. Again, the leakage is in the form of a thin jet flowing along the slope. Despite this apparent similarity between the present study and Chapman and Brink (1987), it is believed that the two wall-induced jets are not identical. While the linear study of Chapman and Brink is reversible (i.e. a cyclone will suck fluid from the right side of the shelf rather than leaking on the left) our nonlinear interaction is not reversible. Indeed, in our case, jets with negative flows are only possible on the left (lower panel of Figure 5); furthermore, they must correspond to a *leakage* rather than entrainment.

## 7. SUMMARY

Before summarizing our results, it is appropriate to stress again that our solution is barotropic, inviscid and hydrostatic. However, the motions associated with the interaction are not constrained to be quasi-geostrophic. The conclusions can be summarized as follows:

- 1) The eddy-boundary interaction is formulated in terms of a thin wall that "cuts" a round vortex at  $t=0$  (Figures 1 and 2). The initial vortex consists of a patch of fluid with *uniform* vorticity; the orbital speed is  $-R_0 fr$ , where  $R_0$ , the Rossby number, is considerably smaller than unity. The edge of the vortex corresponds to a vortex sheet. The nondimensional distance between the wall and the eddy undisturbed edge,  $\epsilon$ , is of the same order as  $R_0$ .
- 2) As a result of the wall which was inserted into the vortex, part of the circulation within the eddy is blocked. To compensate for this partial blocking, the (eddy) flow adjacent to the wall intensifies and some fluid is diverted to section 3 (Figure 3).
- 3) For  $t \sim O(f^{-1})$ , the problem can be considered steady and the

eddy can be taken to be stationary. At larger times [ $O(\epsilon f)^{-1}$ ], however, the translation along the wall becomes important and the problem is time-dependent.

4) Analytical solutions are constructed by integrating the continuity and momentum equation over the area shown in Figure 3, and using a perturbation scheme in  $\epsilon$ , the nondimensional penetration of the wall.

5) It is found that the thickness ( $\delta$ ) of the boundary current formed within the vortex near the wall (Figure 4) must be  $O(\epsilon^{1/2}r_0)$ . This results from the relationships  $\bar{u}_2 \sim O(\epsilon f r_0)$ ,  $\int_0^\delta u'_2 d\xi \sim O(\int_0^{r_0} u'_3 d\xi) \sim O(\epsilon^2 f r_0^2)$  and  $\int_0^\delta \bar{u}_2 u'_2 d\xi \sim O(\int_0^{r_0} (u'_3)^2 d\xi) \sim O(\epsilon^3 f^2 r_0^3)$  which imply that  $\delta \sim O(\epsilon^{1/2}r_0)$  and  $u'_2 \sim O(\epsilon^{3/2} f r_0)$ . Here, the bar and prime correspond to the basic state and perturbed state; the subscripts correspond to the section (Figure 3) and  $\gamma^* \sim O(\epsilon)$ . Note that the above scales are valid even though the leaked flow width  $\gamma$  is  $O(\epsilon r_0)$  (see Figure 3).

6) The above conclusions are applicable for both cyclonic and anti-cyclonic vortices. Cyclonic vortices leak on their left side (looking off-shore) whereas anticyclonic eddies leak on their right (Figure 5). The width of the jet corresponding to the leaked flow is  $\epsilon/3$  and the leaked mass flux is  $\frac{1}{3}\epsilon R_0 f r_0^2 H$  (where  $H$  is the fluid depth).

7) As  $t \rightarrow O(1/\epsilon f)$ , our solution breaks down because the problem becomes time dependent. It is expected that, ultimately, the leakage will be terminated as shown in Figure 6.

The above nonlinear theory has applications to various oceanic situations. Due to the  $\beta$ -induced westward translation, almost all oceanic eddies interact with meridional boundaries. However, since the model is barotropic, a direct and quantitative application to ocean eddies such as Gulf Stream rings is impossible. Extension of the model to baroclinic flow, which is more applicable to such cases, is presently under investigation and will be reported elsewhere.

Although a quantitative comparison with observations is difficult at this stage, a qualitative comparison with laboratory experiments and numerical models is certainly possible. Such comparisons were made and a general agreement between the various studies has been noted. For example, it is shown that the laboratory experiments of

Agnon (1986) identified similar leakages of cyclonic rings (Figure 7). The numerical experiments of Chapman and Brink (1987) also indicate that eddies leak fluid as they interact with boundaries; however, their mechanism is probably quite different from that proposed by our new study.

### Acknowledgements

This study began when the author was a guest investigator at the 1986 Summer Geophysical Fluid Dynamics Program at Woods Hole Oceanographic Institution. Discussions with my colleagues M. E. Stern, G. Flierl and W. Dewar were rather helpful. The study was completed at Florida State University with the support of Office of Naval Research contract number N00014-82-C-0404, NSF grant number OCE871103, and The Florida State University. Two anonymous reviewers provided useful comments on an earlier version of this paper.

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